# 6. Transient Conduction

In this lecture we will deal with the conduction heat transfer problem as a time dependent problem in order to investigate the heat transfer behavior with time. Similarly as in the steady state conduction our aim is to obtain the temperature distribution and heat rate through our field of study and this could be obtained using the same procedure followed for the steady state conditions, we have to solve the appropriate form of heat equation and also for simplicity we may use some approach for simpler cases as we are going to discuss.

## 6.1 The Lumped Capacitance Method

The approach of the lumped capacitance method is based on the assumption that the temperature gradient across the media is small. For example, consider a spoon at initial temperature $T_i$ then it is suddenly immersed in a cup of hot tea at temperature $T_\infty$ which is higher than $T_i$, if the spoon immersion starts at time t = 0 the temperature of the solid will increase as t > 0 until at some time it reaches $T_\infty$, this increase in temperature is due to convection heat transfer at the solid – liquid (spoon – Tea) interface. The lumped capacitance method is based on the assumption that the temperature is spatially uniform at any instant during the transient process so that the temperature gradient maybe considered negligible.

$$k \equiv -\frac{q''_x}{(\partial T / \partial x)} \qquad (6.1)$$

The above form of Fourier's law implies that the thermal conductivity is infinite at approximately zero temperature gradient.

After the pervious assumption it is no longer possible to consider the problem from the framework of the heat equation, therefore to obtain the transient temperature response we should apply the overall energy balance on the solid (spoon) and this balance connect the heat gained by the increase in internal energy. Through the equation

$$\dot{E}_{in} = \dot{E}_{st} \qquad (6.2)$$

The energy flowing into the solid (spoon) is due to convection as we mentioned before and it is expressed as

$$q_{conv} = hA_s(T - T_\infty) \qquad (6.3)$$

$$E_{st} = \rho Vc\frac{dT}{dt} \qquad (6.4)$$

Then by equating the last two equations we get

$$hA_s(T - T_\infty) = \rho Vc\frac{dT}{dt} \qquad (6.5)$$

Introducing the temperature difference θ

$$\theta = T_\infty - T$$

Differentiating the above equation we get that dT/dt = - dθ/dt, Substituting in Equation 6.5 and rearranging the equation we obtain

$$\theta = \frac{\rho V c}{h A_s}\frac{d\theta}{dt} \qquad (6.6)$$

By separation of variables and integration from the initial condition at t = 0 and T (0) = $T_i$, we get

$$\frac{\rho V c}{h A_s}\int_{\theta_i}^{\theta}\frac{d\theta}{\theta} = \int_0^t dt \qquad (6.7)$$

$$\therefore \frac{\rho V c}{h A_s}\ln\frac{\theta}{\theta_i} = t \qquad (6.8)$$

Where $\theta_i = T_\infty - T_i$

The fraction $\theta/\theta_i$ can be obtained by rearranging Equation 6.8 such as

$$\frac{\theta}{\theta_i} = \frac{T_\infty - T}{T_\infty - T_i} = \exp\left[\left(\frac{h A_s}{\rho V c}\right)t\right] \qquad (6.9)$$

Equation 6.9 could be easily used to determine the temperature T reached after a given time t or the time t needed to reach a certain temperature T. The coefficient of t inside the exponential function may be regarded as a thermal time constant. Hence, the thermal time constant can be defined as

$$\tau_t = \left(\frac{1}{h A_s}\right)(\rho V c) = R_t C_t \qquad (6.10)$$

Where $R_t$ is the convection resistance to heat transfer and $C_t$ is the lumped thermal capacitance of the solid. This new term is an indication to the material response to the change in thermal environment. As this constant increases the response time of the material to thermal changes decreases. An analogy between these types of heat conduction problem can be made. This analogy is shown in Figure 6.1 below.

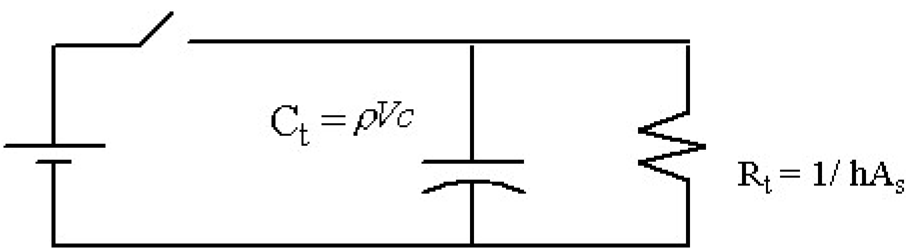


Figure 6.1 Analogous Electric circuit to a transient heat conduction problem

The amount of heat transferred from time t = 0 till a certain time t can be calculated by integrating the convection heat transferred from time t = 0 till a certain time t. this is simply shown in Equation 6.11 as

$$Q = \int_0^t q\, dt = hA\int_0^t \theta\, dt \qquad (6.11)$$

Substituting for θ from Equation 6.9 and performing the integrating Equation 6.11 yields

$$Q = (\rho Vc)\theta_i\left[1 - \exp\left(\frac{t}{\tau_t}\right)\right] \qquad (6.12)$$

The amount of heat transferred Q is that taken or gained by the internal energy. So that Q is positive when the solid is heated and there is a gain in the internal energy and Q is negative when the solid is cooled and there is a decrease in the internal energy.

## 6.2 Validity of the Lumped Capacitance Method

Lumped capacitance method is very desirable due to its simplicity and convenience. However, it is important to determine under what conditions it should be used in order to yield reasonable accuracy.

To develop a suitable criterion consider a steady state conduction through the plane wall of area A, as seen in Figure 6.2, where one surface is maintained at temperature $T_{s,1}$ and the other is exposed to a fluid of temperature $T_\infty$ where $T_{s,1} > T_\infty$. The surface temperature, $T_{s,2}$, has a value between $T_{s,1}$ and $T_\infty$ is called, thus under steady state conditions the surface energy balance will be

$$\frac{kA}{L}\left(T_{s,1} - T_{s,2}\right) = hA(T_{s,2} - T_\infty) \qquad (6.13)$$

Rearranging the Equation 6.13 we get

$$\frac{T_{s,1} - T_{s,2}}{T_{s,2} - T_\infty} = \frac{(L/kA)}{(1/hA)} = \frac{R_{cond}}{R_{conv}} = \frac{hL}{k} \equiv Bi \qquad (6.14)$$

The quantity (hL/k) appearing in Equation 6.14 is a dimensionless parameter called Biot number. This number plays a fundamental role in conduction problems involving surface convection effects. If Biot number is less than unity, then the conduction resistance within the solid is much less than the convection resistance across the boundary layer. Thus the assumption of a uniform temperature distribution is reasonable. Hence, the lumped capacitance method could only be used if:

$$Bi = \frac{hL_c}{k} < 0.1 \qquad (6.15)$$

Where $L_c$ is the characteristic length of the solid shape and for more complicated shapes it can be defined as $L_c \equiv V/A_s$ for simplicity and the characteristic length $L_c$ is reduced to L for a plane wall of thickness 2L and to $r_o/2$ for a long cylinder and $r_o/3$ for a sphere.

When substituting with the characteristics length $L_c \equiv V/A_s$ in the exponent of Equation 6.9 we get the following simplification

$$\frac{hA_s t}{\rho Vc} = \frac{h\, t}{\rho\, c\, L_c} = \frac{hL_c}{k}\frac{k}{\rho c}\frac{t}{L_c^2} = \frac{hL_c}{k}\frac{\alpha t}{L_c^2} \qquad (6.16)$$

Or

$$\frac{hA_s t}{\rho V c} = Bi \cdot Fo \qquad (6.17)$$

Where a new parameter, called Fourier number, is introduced which is a dimensionless time

$$Fo \equiv \frac{\alpha t}{L_c^2} \qquad (6.18)$$

Substituting from Equation 6.17 into Equation 6.9 we get

$$\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = \exp(-Bi \cdot Fo) \qquad (6.19)$$

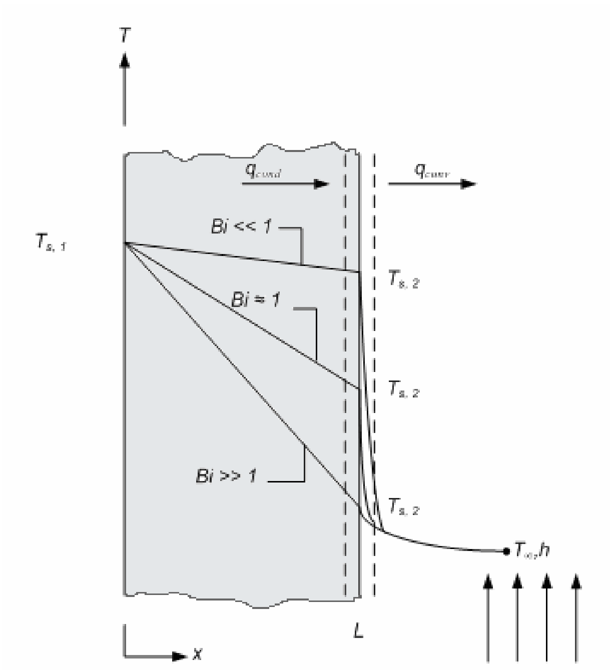


Figure 6.2 Biot number effect on steady state temperature distribution

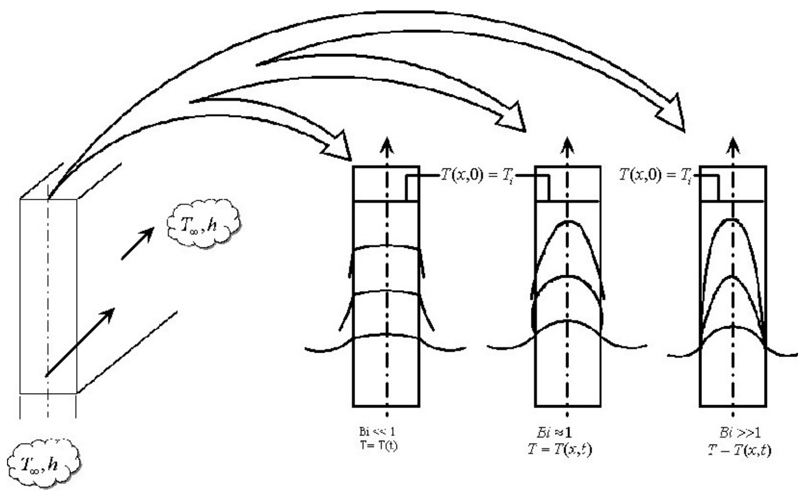

Figure 6.3 Transient temperature distribution for different Bi numbers in a plane symmetrically cooled from the two sides by convection

**Exercise 6.1:** A thermocouple junction, who may be approximated as a sphere, is to be used for temperature measurement of cooling air stream for electronic box. The convection coefficient between the junction and the air is known to be h = 400 W / $m^2$ K. The junction thermophysical properties are:

k = 20 W / m · K
c = 400 J / kg · K
ρ = 8500 kg / $m^3$

Determine the junction diameter needed for the thermocouple to have a time constant of 1 s.
If the junction is at 25 ºC and is placed in air at 15 ºC, how long would it take for the junction to reach 16 ºC?

## 6.3 Spatial Effect

The transient conduction problem in its general form is described by the heat equation either in Cartesian, cylindrical or spherical coordinates. Many problems such as plane wall needs only one spatial coordinate to describe the temperature distribution, with no internal generation and constant thermal conductivity the general heat equation has the following form

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad (6.20)$$

Equation 6.20 is a second order in displacement and first order in time; therefore we need an initial condition and two boundary conditions in order to solve it. Following are some graphical solutions for simple cases.

### 6.3.1 Large Plate of Finite Thickness Exposed to Convection

In this case the initial condition is:

$$T(x,0) = T_i$$

And the boundary conditions are

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0$$

And

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=L} = h[T(L,t) - T_\infty]$$

Dimensional analysis is a very useful method when the problem involves many variables as it reduces the variables being involved into groups and provides mathematical relations between them

For this reason, two dimensional groups are used in the graphical solution of the one dimensional transient conduction, the two groups are the dimensionless temperature difference $\theta^* = \theta / \theta_i$ and the dimensionless time Fourier number $t^* = Fo = \alpha\, t / L_c^2$ and the dimensionless displacement $x^* = x / L$. Solutions are represented in graphical forms that illustrate the functional dependence of the transient temperature distribution on the Biot and Fourier numbers.

Figure 6.4 may be used to obtain the midplane temperature of the wall, T (0, t) ≡ $T_a$ (t), at any time during the transient process.

If $T_o$ is known for particular values of Fo and Bi, Figure 6.5 may be used to determine the corresponding temperature at any location off the midplane. Hence Figure 6.5 must be used in conjunction with Figure 6.4. For example, if one wishes to determine the surface temperature ($x^* = \pm 1$) at some time t, Figure 6.4 would first be used to determine $T_o$ at t. Figure 6.5 would then be used to determine the surface temperature from the knowledge of $T_o$. The procedure would be rolled back if the problem involves determining the time required for the surface to reach a prescribed temperature.

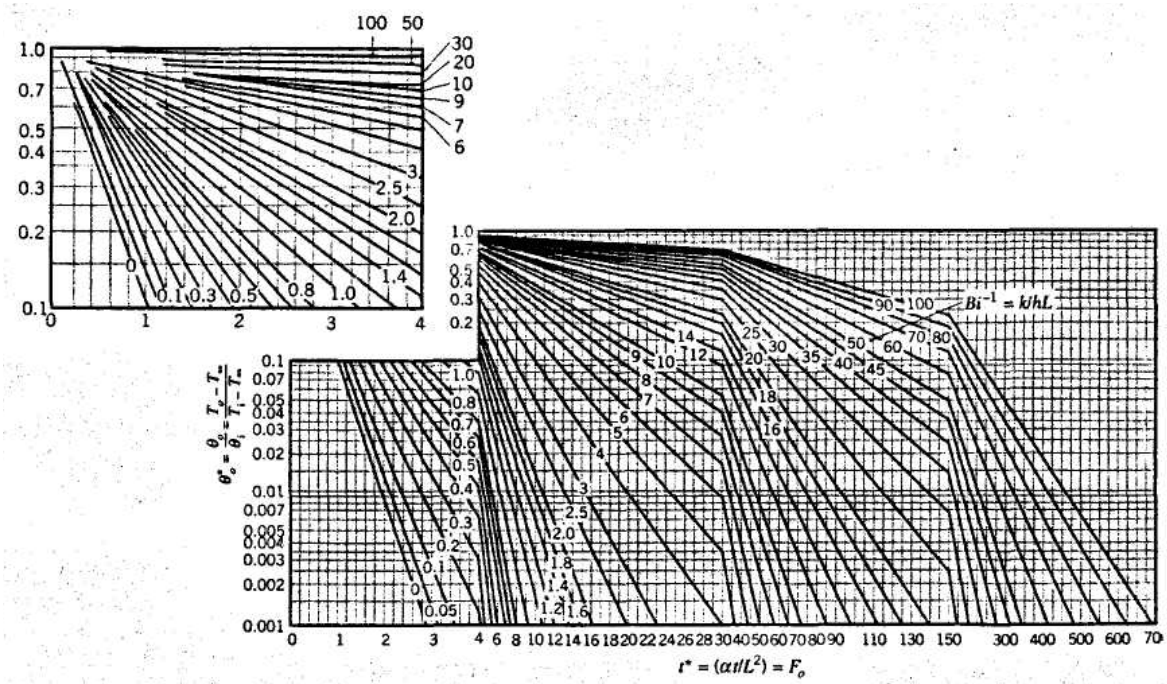


**Figure 6.4 Midplane temperature as a function of time for a plane wall of thickness 2L**

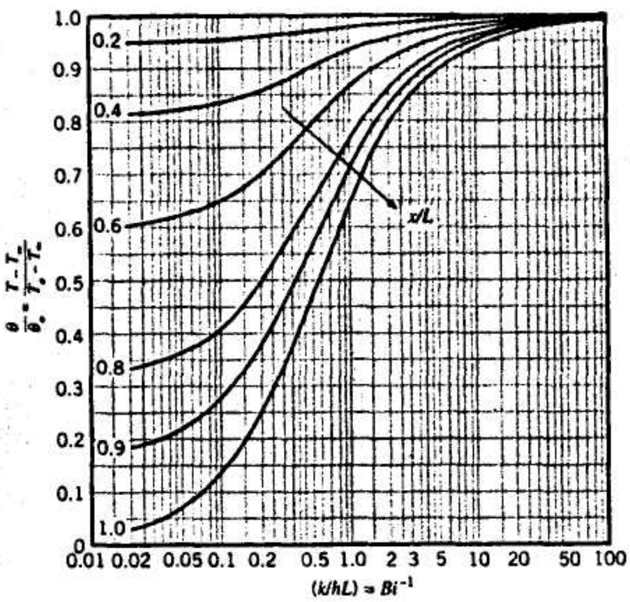


**Figure 6.5 Temperature distribution in a plane wall of thickness 2L**

Graphical results for the energy transferred from a plane wall over the time interval t are presented in Figure 6.6. The dimensionless energy transfer $Q/Q_o$ is expressed exclusively in terms of Fo and Bi.

The foregoing charts may also be used to determine the transient response of a plane wall, an infinite cylinder, or a sphere subjected to a sudden change in surface temperature. For such a condition it is only necessary to replace $T_\infty$ by the prescribed surface temperature $T_{s,}$ and to set $Bi^{-1}$ equal to zero. In so doing, the convection coefficient is tacitly assumed to be infinite, in which case $T_\infty = T_s$.

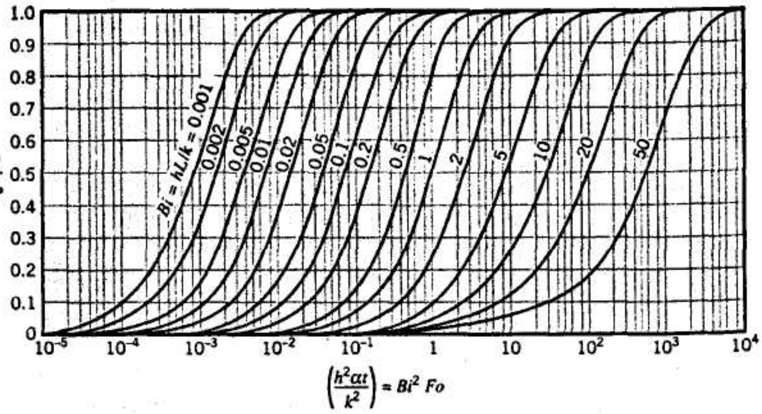


**Figure 6.6 Internal energy change as a function of time for plane wall of thickness 2L**

Similarly for the infinite cylinder the results are presented in Figures 6.6 to 6.8 only the $L_c$ is replaced by $r_o$ then $Bi^{-1} = k / h\, r_o$.

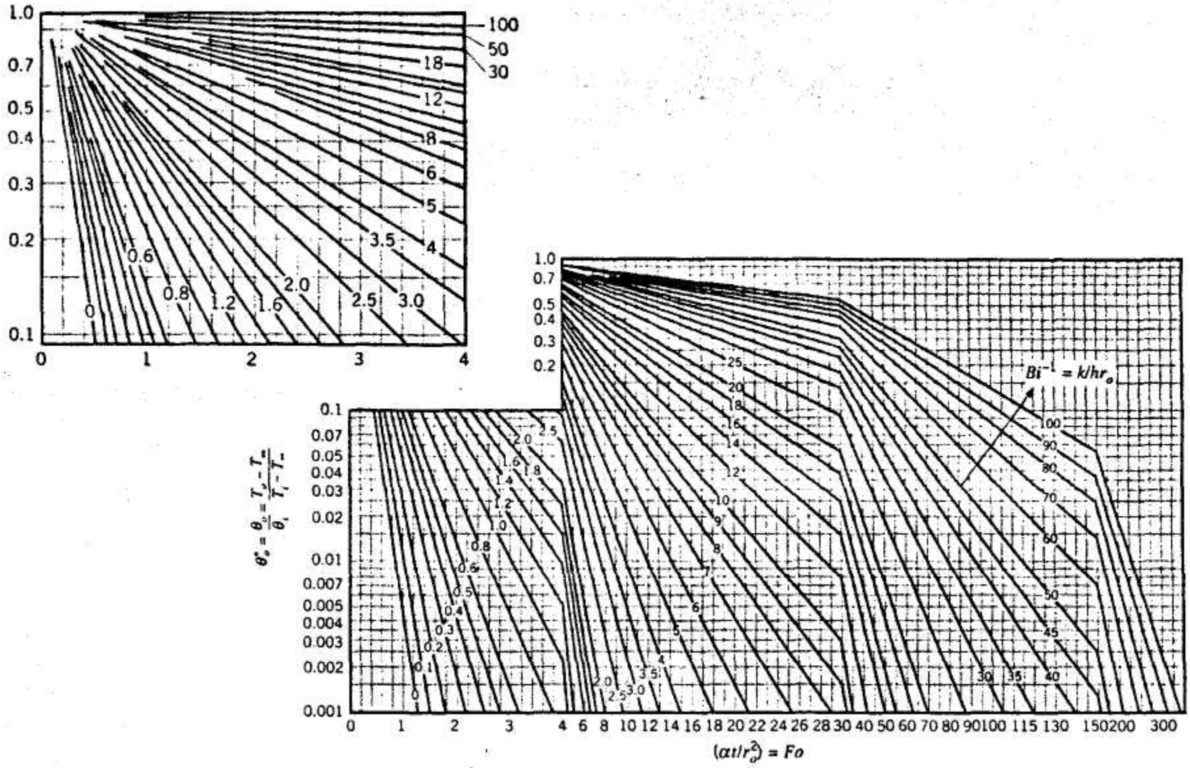


**Figure 6.6 Centerline temperature as a function of time for an infinite cylinder of radius $r_o$**

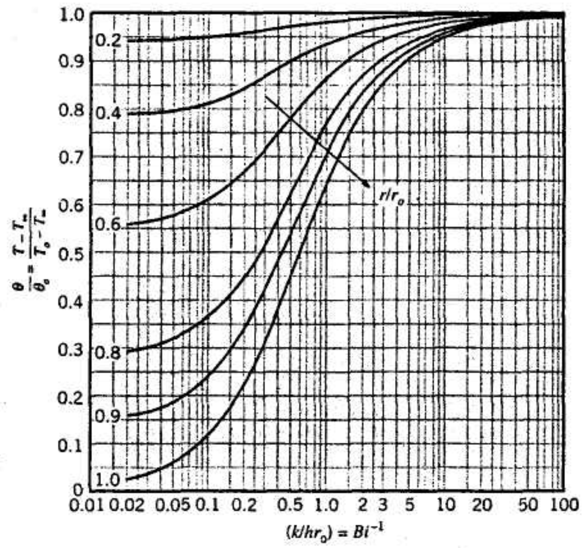


**Figure 6.7 Temperature distribution in an infinite cylinder of radius $r_o$**

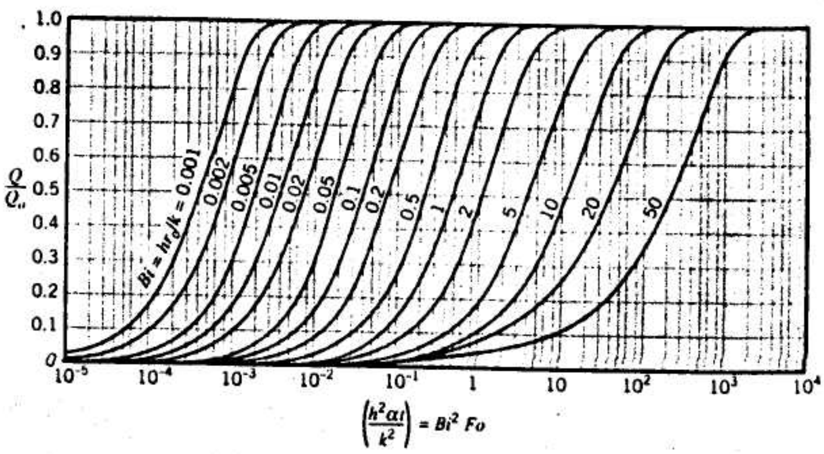


**Figure 6.8 Internal energy change as a function of time for an infinite cylinder of radius $r_o$**

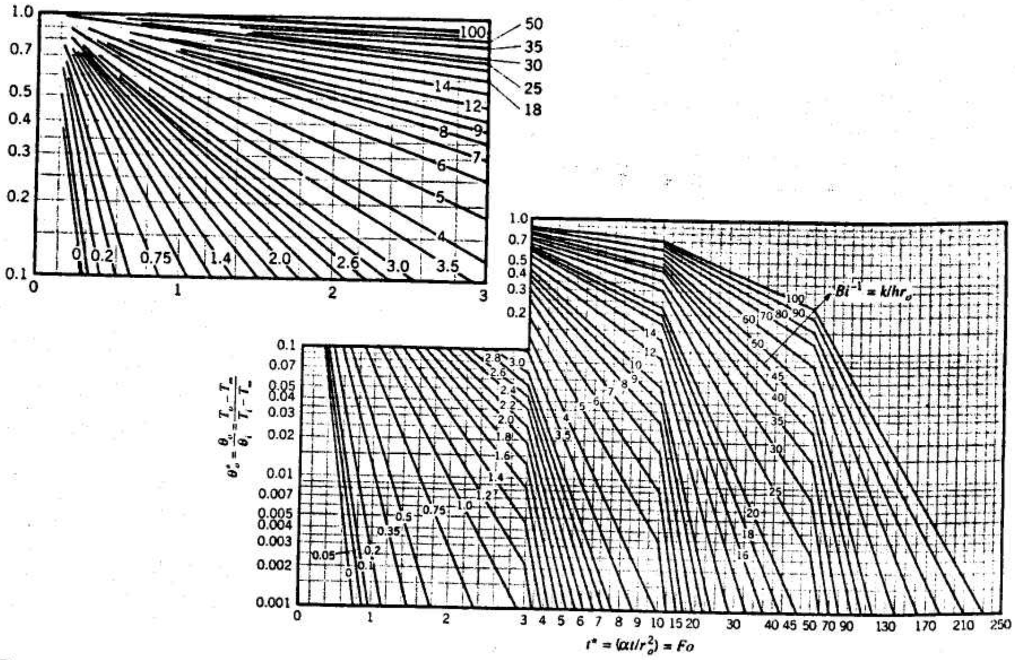

**Figure 6.9 Center temperature as a function of time sphere of radius $r_o$**

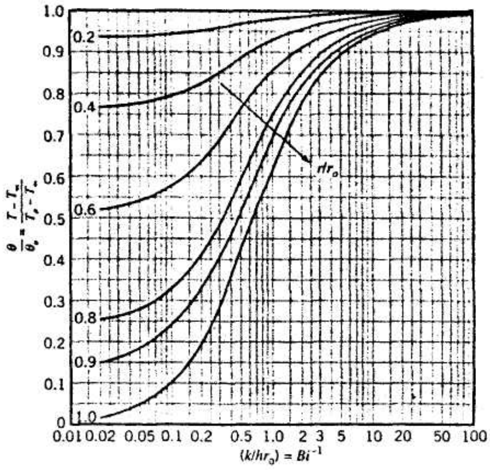

**Figure 6.10 Temperature distribution in a sphere of radius $r_o$**

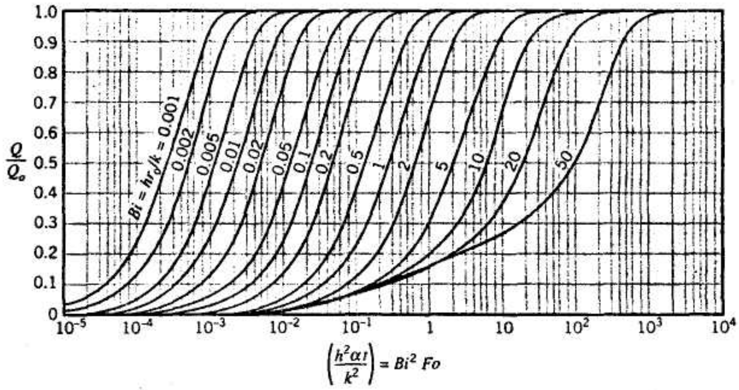

**Figure 6.11 Internal energy change as a function of time for a sphere of radius $r_o$**

## 6.4 Semi-Infinite Solids

Another simple geometry for which analytical solutions may be obtained is the semi-infinite solids it is characterized by a single defined surface since it extends to infinity in all directions except one. If a sudden change is imposed to this surface transient one dimensional conduction will occur within the solid.

Equation 6.20 still applies as a heat equation. Under the same assumptions which is one dimensional with no heat generation heat transfer.

The initial condition is

$$T(x,0) = T_i$$

While the interior boundary condition is

$$T(x \to \infty, t) = T_i$$

for the above initial and interior boundary conditions three closed form solutions have been obtained for the surface conditions, which are applied suddenly to the surface at t = 0. These conditions include application of constant surface temperature, constant heat flux and exposure of a surface to a fluid characterized by $T_\infty \neq T_i$ and the convection coefficient h, as shown in Figure 6.12.

For each case, an analytical solution can be obtained as:

Case (1) Constant surface temperature: $T(0,t) = T_i$

$$\frac{T(x,t) - T_i}{T_i - T_s} = erf\left(\frac{x}{2\sqrt{\alpha t}}\right) \quad (6.21)$$

$$q_s''(t) = \frac{k(T_s - T_i)}{\sqrt{\pi \alpha t}} \quad (6.22)$$

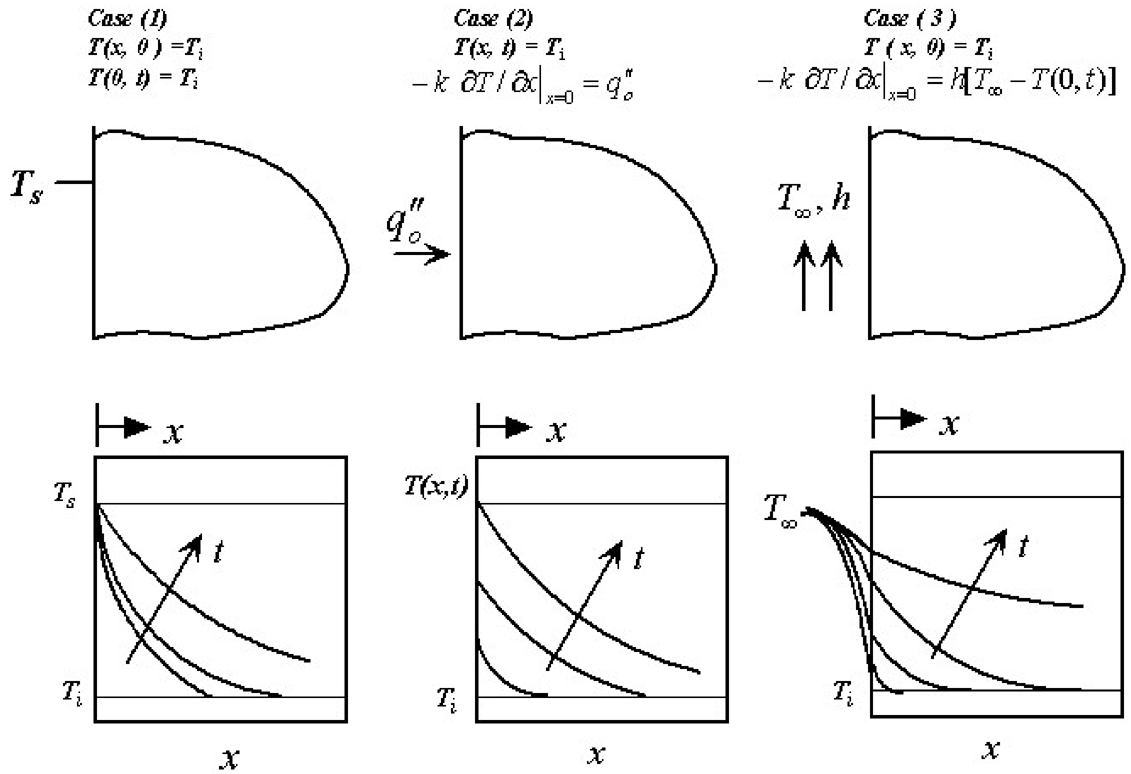


**Figure 6.12 Transient Temperature distribution in a semi-infinite solid for three surface conditions: case (1) constant surface temperature, case (2) constant surface heat flux, and case (3) surface convection**

Case (2) Constant surface heat flux: $q''_s = q''_o$

$$T(x,t) - Ti = \frac{2q''_o(\alpha t/\pi)^{1/2}}{k}\exp\left(\frac{-x^2}{4\alpha t}\right) - \frac{q''_o x}{k}\, erfc\left(\frac{x}{2\sqrt{\alpha t}}\right) \qquad (6.23)$$

Case (3) Surface convection: $-k\ \partial T/\partial x|_{x=0} = h[T_\infty - T(0,t)]$

$$\frac{T(x,t) - T_i}{T_\infty - T_i} = erfc\,(\frac{x}{2\sqrt{\alpha t}}) - \left[\exp\left(\frac{hx}{k} + \frac{h^2\alpha t}{k^2}\right)\right]\left[erfc\left(\frac{x}{2\sqrt{\alpha t}} + \frac{h\sqrt{\alpha t}}{k}\right)\right] \qquad (6.24)$$

Where the complementary error function erfc w is defined as erfc w = 1- erf w.